A NOTE ON SOME MICROLOCAL ESTIMATES USED TO PROVE THE CONVERGENCE OF SPLITTING METHODS RELYING ON PSEUDO-SPECTRAL DISCRETIZATIONS.

JOACKIM BERNIER, FERNANDO CASAS, AND NICOLAS CROUSEILLES

Abstract. In [BCC20], we used some classical microlocal estimates to prove the convergence of our splitting methods (for example page A671). In this note, through Corollary 2 and Remark 1, we provide a detailed proof of these estimates. All the proofs rely on results presented in [NR10].

We consider the classes of symbols $S(\langle x\rangle^{s_1}+\langle \xi\rangle^{s_2};\langle \xi\rangle,\langle x\rangle),\ s_1,s_2\in\mathbb{R}$. By definition (see Definition 1.1.1 page 19 in [NR10]), it contains the symbols $a(x,\xi)$ such that

$$\forall \gamma,\delta\in\mathbb{N},\ |\partial_x^\gamma\partial_\xi^\delta a(x,\xi)|\lesssim_{\gamma,\delta}(\langle x\rangle^{s_1}+\langle \xi\rangle^{s_2})\langle x\rangle^{-\gamma}\langle \xi\rangle^{-\delta}.$$

We are going to prove the following proposition.

Proposition 1. *If $s_1,s_2\geq 0$ and $b\in S(\langle x\rangle^{s_1}+\langle \xi\rangle^{s_2};\langle \xi\rangle,\langle x\rangle)$ then*

$$\forall u\in\mathscr{S}(\mathbb{R}^d),\ |(u,b^w u)_{L^2}|\lesssim_{b,s_1,s_2}(u,(\langle x\rangle^{s_1}+\langle \xi\rangle^{s_2})^w u)_{L^2}.$$

This proposition is useful to get the following corollaries.

Corollary 1. *If $s\geq 0$ and $a\in S(\langle x\rangle^{s}+\langle \xi\rangle^{s};\langle \xi\rangle,\langle x\rangle)$ then*

$$\forall u\in\mathscr{S}(\mathbb{R}^d),\ \|a^w u\|_{L^2}\lesssim_{a,s}\|u\|_{X^s}$$

where

$$\|u\|_{X^s}^2:=\int_{\mathbb{R}^d}\langle x\rangle^{2s}|u(x)|^2\mathrm{d}x+\int_{\mathbb{R}^d}\langle \xi\rangle^{2s}|\hat u(\xi)|^2\mathrm{d}\xi.$$

Proof of Corollary 1. Since we have

$$(\langle x\rangle^{s}+\langle \xi\rangle^{s})^2\leq 2(\langle x\rangle^{2s}+\langle \xi\rangle^{2s}),$$

applying the Proposition 1.2.9 page 29 and the Theorem 1.2.16 page 31 in [NR10], we get a symbol

$$c\in S((\langle x\rangle^{s}+\langle \xi\rangle^{s})^2;\langle \xi\rangle,\langle x\rangle)\subset S(\langle x\rangle^{2s}+\langle \xi\rangle^{2s};\langle \xi\rangle,\langle x\rangle),$$

such that

$$c^w=(a^w)^*a^w.$$

Consequently, applying Proposition 1, for $u\in\mathscr{S}(\mathbb{R}^d)$, we have

$$\|a^w u\|_{L^2}^2=(u,c^w u)_{L^2}\leq c_{a,s}(u,(\langle x\rangle^{2s}+\langle \xi\rangle^{2s})^w u)_{L^2}=c_{a,s}\|u\|_{X^s}^2$$

where $c_{a,s}$ is a constant depending only on a and s. ☐

J.B. thanks Paul Alphonse for his help and his advices to write this note.

Corollary 2. *If $s \geq 0$ and $\alpha, \beta > 0$ are such that $\alpha + \beta \leq s$ then*

$$\forall u \in \mathscr{S}(\mathbb{R}^d), \ \|\langle x\rangle^\alpha \langle\nabla\rangle^\beta u\|_{L^2} \lesssim_{a,s} \|u\|_{X^s}.$$

Proof of Corollary 1. Since, by Young, we have

$$\langle x\rangle^\alpha \langle\xi\rangle^\beta \leq \frac{\alpha}{\alpha+\beta}\langle x\rangle^{\alpha+\beta} + \frac{\beta}{\alpha+\beta}\langle\xi\rangle^{\alpha+\beta} \leq \langle x\rangle^s + \langle\xi\rangle^s,$$

applying the Theorem 1.2.16 page 31 in [NR10], we get a symbol

$$a \in S(\langle x\rangle^\alpha\langle\xi\rangle^\beta; \langle\xi\rangle, \langle x\rangle) \subset S(\langle x\rangle^s + \langle\xi\rangle^s; \langle\xi\rangle, \langle x\rangle),$$

such that

$$a^w = \langle x\rangle^\alpha \langle\nabla\rangle^\beta.$$

Consequently, we conclude by applying the Corollary 1. □

Remark 1. This corollary could be easily extended to control the terms we had in [BCC20]. For exemple, provided that $\alpha+\beta+\gamma \leq s$, we could also control things like $\|\langle\nabla\rangle^\gamma\langle x\rangle^\alpha\langle\nabla\rangle^\beta u\|_{L^2(\mathbb{R}^d)}$ or $\|\langle\partial_{x_1}\rangle^\gamma\langle x_2\rangle^\alpha\langle\partial_{x_2}\rangle^\beta u\|_{L^2(\mathbb{R}^d)}$.

The proof of the Proposition 1 relies on the following technical lemma whose proof is given in the subsection 1.1 of the Appendix.

Lemma 1. *If $a \in S(\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2}; \langle\xi\rangle, \langle x\rangle)$, there exists $r_1 \in S(\langle x\rangle^{s_1-1} + \langle\xi\rangle^{s_2}; \langle\xi\rangle, \langle x\rangle)$, $r_2 \in S(\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2-1}; \langle\xi\rangle, \langle x\rangle)$ such that*

$$A_a = a^w + r_1^w + r_2^w$$

where A_a is the Anti-Wick operator with symbol a (see Definition 1.7.3 page 53 in [NR10]*).*

Finally, we focus on the proof of the Proposition 1.

Proof of Proposition 1. Let $s_1, s_2 \geq 0$ and $b \in S(\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2}; \langle\xi\rangle, \langle x\rangle)$. We aim at proving that

$$\forall u \in \mathscr{S}(\mathbb{R}^d), \ |(u, b^w u)_{L^2}| \lesssim_{b,s_1,s_2} (u, (\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2})^w u)_{L^2}. \tag{1}$$

Since $b \in S(\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2}; \langle\xi\rangle, \langle x\rangle)$, there exists a constant $c_b > 0$ such that

$$\forall x, \xi \in \mathbb{R}^d, \ a(x,\xi) := c_b(\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2}) \pm b(x,\xi) \geq 0.$$

Since it is clear that $\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2} \in S(\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2}; \langle\xi\rangle, \langle x\rangle)$, we have $a \in S(\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2}; \langle\xi\rangle, \langle x\rangle)$ and so, by applying the Lemma 1, we get $r_1 \in S(\langle x\rangle^{s_1-1} + \langle\xi\rangle^{s_2}; \langle\xi\rangle, \langle x\rangle)$ and $r_2 \in S(\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2-1}; \langle\xi\rangle, \langle x\rangle)$ such that

$$A_a = a^w + r_1^w + r_2^w.$$

Since the symbol a is nonnegative, by applying the Proposition 1.7.6 page 53 in [NR10], we know that A_a is a nonnegative operator and so

$$(u, A_a u)_{L^2} = c_b(u, (\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2})^w u)_{L^2} \pm (u, b^w u)_{L^2} + (u, r_1^w u)_{L^2} + (u, r_2^w u)_{L^2} \geq 0. \tag{2}$$

Finally, we have to control $(u, r_1^w u)_{L^2}$ and $(u, r_2^w u)_{L^2}$. By symmetry, we only focus on $(u, r_1^w u)_{L^2}$.

We proceed by induction.

• *Case* $s_1 > 1$. By the induction assumption, we know that

$$(u, r_1^w u)_{L^2} \lesssim_{b,s_1,s_2} (u, (\langle x\rangle^{s_1-1} + \langle \xi\rangle^{s_2})^w u)_{L^2},$$

and so since $\langle x\rangle^{s_1-1} \leq \langle x\rangle^{s_1}$ we have

$$(u, r_1^w u)_{L^2} \lesssim_{b,s_1,s_2} (u, (\langle x\rangle^{s_1} + \langle \xi\rangle^{s_2})^w u)_{L^2}.$$

• *Case* $s_1 \leq 1$. Applying Theorem 1.2.16 page 31 in [NR10], we get a symbol $f \in S(\langle\xi\rangle^{-s_2}\langle x\rangle^{s_1-1} + 1; \langle\xi\rangle, \langle x\rangle) \subset S(1;1,1)$ such that

$$f^w = \langle\nabla\rangle^{-s_2/2} r_1^w \langle\nabla\rangle^{-s_2/2}.$$

Consequently, applying the Calderón-Vaillancourt theorem (Theorem 1.7.14 page 58 in [NR10]), this operator is bounded in L^2 and so

$$\forall u \in \mathscr{S}(\mathbb{R}^d),\ (r_1^w \langle\nabla\rangle^{-s_2/2} u, \langle\nabla\rangle^{-s_2/2} u)_{L^2} \lesssim_{r_1^w, s_2} \|u\|_{L^2}^2.$$

As a consequence, the change of function $u \leftarrow \langle\nabla\rangle^{-s_2/2} u$ provides the estimate

$$\forall u \in \mathscr{S}(\mathbb{R}^d),\ (r_1^w u, u)_{L^2} \lesssim_{r_1,s_2} (u, \langle\nabla\rangle^{s_2} u)_{L^2} \lesssim_{r_1,s_2} (u, (\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2})^w u)_{L^2}.$$

In any case, we have

$$(r_1^w u, u)_{L^2} + (r_2^w u, u)_{L^2} \lesssim_{b,s_1,s_2} (u, (\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2})^w u)_{L^2}.$$

Plugging this estimate in (2) provides naturally the estimate (1) we aimed at proving.

□

1. Appendix

1.1. **Proof of Lemma 1.** We aim at proving that if $a \in S(\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2}; \langle\xi\rangle, \langle x\rangle)$ then there exists $r_1 \in S(\langle x\rangle^{s_1-1} + \langle\xi\rangle^{s_2}; \langle\xi\rangle, \langle x\rangle)$, $r_2 \in S(\langle x\rangle^{s_1} + \langle\xi\rangle^{s_2-1}; \langle\xi\rangle, \langle x\rangle)$ such that

$$A_a = a^w + r_1^w + r_2^w.$$

Applying the Proposition 1.7.9 page 55, we know that

$$A_a = b^w$$

where

$$b(x,\xi) := 2^d \int_{\mathbb{R}^d} a(y,\eta) e^{-|x-y|^2 - |\xi-\eta|^2} \frac{\mathrm{d}y}{(2\pi)^{d/2}} \frac{\mathrm{d}\eta}{(2\pi)^{d/2}}. \tag{3}$$

Consequently, we just have to decompose b in the good classes.

Naturally, the Taylor expansion at the order 1 of a in (x,ξ) is

$$a(y,\eta) = a(x,\xi) + \int_0^1 \partial_x a(x + t(y-x), \xi + t(\eta-\xi)) \mathrm{d}t (y-x) \\ + \int_0^1 \partial_\xi a(x + t(y-x), \xi + t(\eta-\xi)) \mathrm{d}t (\eta-\xi).$$

Plugging this expansion in (3) and realizing the change of coordinate $(y,\eta) \leftarrow (y-x, \eta-\xi)$, we are naturally led to set

$$r_1(x,\xi) = 2^d \int_{\mathbb{R}^d} \int_0^1 \partial_x a(x+ty, \xi+t\eta)\ y\ e^{-|y|^2-|\eta|^2} \mathrm{d}t \frac{\mathrm{d}y}{(2\pi)^{d/2}} \frac{\mathrm{d}\eta}{(2\pi)^{d/2}}$$

and

$$r_2(x,\xi) = 2^d \int_{\mathbb{R}^d}\int_0^1 \partial_\xi a(x+ty,\xi+t\eta)\ \eta\ e^{-|y|^2-|\eta|^2}\mathrm{d}t\frac{\mathrm{d}y}{(2\pi)^{d/2}}\frac{\mathrm{d}\eta}{(2\pi)^{d/2}}.$$

By symmetry, we just check that $r_1 \in S(\langle x\rangle^{s_1-1}+\langle \xi\rangle^{s_2};\langle \xi\rangle,\langle x\rangle)$.

By definition, we just have to prove that

$$|\partial_x^\alpha \partial_\xi^\beta r_1(x,\xi)| \lesssim_{\alpha,\beta} (\langle x\rangle^{s_1-1}+\langle \xi\rangle^{s_2})\langle x\rangle^{-\alpha}\langle \xi\rangle^{-\beta}. \tag{4}$$

Since, by assumption, we know that

$$|\partial_x^\gamma \partial_\xi^\delta a(x,\xi)| \lesssim_{\gamma,\delta} (\langle x\rangle^{s_1}+\langle \xi\rangle^{s_2})\langle x\rangle^{-\gamma}\langle \xi\rangle^{-\delta},$$

we deduce that

$$|\partial_x^{\alpha+1} \partial_\xi^\beta a(x+ty,\xi+t\eta)| \lesssim_{\alpha,\beta} (\langle x+ty\rangle^{s_1}+\langle \xi+t\eta\rangle^{s_2})\langle x+ty\rangle^{-\alpha-1}\langle \xi+t\eta\rangle^{-\beta}.$$

Recalling the Peetre's inequality (0.1.2) page 19 :

$$\langle x+y\rangle^s \lesssim_s \langle x\rangle^s\langle y\rangle^{|s|},\ \forall x,y\in\mathbb{R}^d, s\in\mathbb{R},$$

we get

$$\begin{aligned}|\partial_x^{\alpha+1} \partial_\xi^\beta a(x+ty,\xi+t\eta)| &\lesssim_{\alpha,\beta,s_1,s_2} (\langle x\rangle^{s_1}\langle y\rangle^{s_1}+\langle \xi\rangle^{s_2}\langle \eta\rangle^{s_2})\langle x\rangle^{-\alpha-1}\langle \xi\rangle^{-\beta}\langle y\rangle^{\alpha+1}\langle \eta\rangle^{\beta}\\ &\lesssim_{\alpha,\beta,s_1,s_2} (\langle x\rangle^{s_1}+\langle \xi\rangle^{s_2})\langle x\rangle^{-\alpha-1}\langle \xi\rangle^{-\beta}\langle y\rangle^{\alpha+1+s_1}\langle \eta\rangle^{\beta+s_2}.\end{aligned} \tag{5}$$

Finally, observing that

$$(\langle x\rangle^{s_1}+\langle \xi\rangle^{s_2})\langle x\rangle^{-1} \leq \langle x\rangle^{s_1-1}+\langle \xi\rangle^{s_2},$$

plugging (5) in the definition of r_1 yields to (4), i.e. $r_1 \in S(\langle x\rangle^{s_1-1}+\langle \xi\rangle^{s_2};\langle \xi\rangle,\langle x\rangle)$.

References

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Institut de Mathématiques de Toulouse ; UMR5219, Université de Toulouse ; CNRS, Université Paul Sabatier, F-31062 Toulouse Cedex 9, France
E-mail address: joackim.bernier@math.univ-toulouse.fr

Universitat Jaume I, 12071 Castellon, Spain.
E-mail address: fernando.casas@uji.es

Univ Rennes, INRIA, MINGuS team, IRMAR - UMR 6625, F-35042 Rennes, France
E-mail address: nicolas.crouseilles@inria.fr